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SHIFTS AND SEASONALITIES IN ECONOMETRIC RELATIONSHIPS

BY

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Introduction

Perhaps the most distinct difference between modern econometric analysis and classical mothods is the difference of the specification of the role played by time as a variable. Classical methods used to consider time not merely as an independent variable, but as a "cause" of changes in economic magnitudes over time. The result was the development of time series analysis in the direction of difining certain time components and isolating the effects of each component by means of certain time-functions. Thus "trend" used to be considered as a certain function (e. g., a polynomial) in time. Periodic functions were consideted in order to account for the cyclical and seasonal components whenever they are suspected. These three components were held responsible for the systematic behaviour in economic time series. What was left over and above was considered as a random component. No constructive attempt had been made for identifying the parts of these random elements with their major sources, namely, observation errors and errors in behaviour.

The purely temporal explanations were not the only explanations. In many cases attempts were made to identify one or more of the above mentioned components with certain exogenous elements changing over time. Such elements were exogenous in the sense that they originate outside the economic system, thus affecting it without being affected by it. One had therefore to look for meteorological affecting the structural economic relationships. If we then consider time as a dummy shift variable, then there is no reason to cofine the class of shift variables to trend terms alone. Any shift variable which suits the description of the shifts suspected to exist in the relationship can be used. It is our purpose here to study rnother type of shift variables which is capable of taking different levels at different periods of time in a rather discontinuous manner. The problem is highly connected with the treatment of seasonal variations in the specific manner suggsted in the following sections.

Now, if the time-units used are shorter than a year, one has to account for seasonalities. The classical approach would attempt at constructing some simple model which helps to isolate the seasonal elements, either for purposes of estimating their magnitudes, or for their elimination. Such models assume—either explicitly or implicitly—some sort of an assumption on the behaviour of economic variables, and on the relationships between their components. The validity of the estimators of the structural parameters would therefore depend on the validity of the assumptions underlying the method of deseasonalizing the variables, and the consistency between these assumptions and those underlying the specific methods of structural estimation adopted.

It is then but natural to extend our original system of equations so as to take into account seasonal variables. The process of estimation to be adopted would be then that one which conforms with the new model, and not the one which is used with respect to annual data as applied to seasonally adjusted series This approach has been suggested by Prof. Klein(1), but its implications have not been studied in full detail yet. This led Prof. Robert M. Solow to pass the following comment on Klein's approach in his review(2) of Klein's book [1] :

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(1) See [1] L. Klein : A Text book of Econometrics — and [2] L. Klein ; and H. Barger : "A Quarterly Model of the United States Economy" — J. A. S. A., Vol. 49, 1954 pp. 413-37.

(2) See the American Economic Review, 1955 p. 429.

Thus (1) becomes:

$$(1.2) \qquad Y_{ts} = a_o + \sum_{r=1}^{m'} k_r \, v_{rts} + \sum_{i=1}^{p} a_i \, z_{its} + u_{ts}$$

Where v_{rts} are seasonal variables, *i. e.*, variables responsible for seasonal variations in the equation. They are independent of the u's, and are linearly independent of the z's.

(2) The Unrestricted Multiplicative Case:

> "Perhaps other parameters, in addition to the constant term are affected by seasonal change". (ibid.)

Accordingly (1) becomes:

$$(1.3) \qquad Y_{ts} = \sum_{i=o}^{p} \Big(\sum_{r=1}^{m'} k_{ir} \, v_{rts} \Big) \Big(a_i \, z_{its} \Big) + u_{ts}$$

where $z_{ots} = 1$ for all t and s, so that a_o is the constant term.

(3) The Restricted Multiplicative Case:

If in (1.3) we restrict the k's to be the same for all i, we obtain the formula suggested by Klein and Barger [2]. Thus:

$$(1.4) \qquad Y_{ts} = \Big(\sum_{r=1}^{m'} k_r \, v_{rts} \Big) \Big(\sum_{i=o}^{p} a_i \, z_{its} \Big) + u_{tu}$$

In all the above formulations we assume that the u's are non-auto-correlated, and that they are independent of the v's and the z's. The variables v_{rts} can be used whenever enough observations are available. However, the usual case is that information on such variables is lacking, and we shall assume, as Klein did, that they are shift (dummy) variables given by:

$$(1.5) \qquad v_{rts} = v_{rs} = \delta_{rs} \qquad (r,s = 1, \ldots, m)$$

since $T = \Sigma n_i$. Thus the matrix in (2.2) is nonsingular. This is due to the fact that in any r-th segment of a year, the v's satisfy the homogeneous equation:

$$(2.3) \qquad \sum_i v_{ir} - \sum_i n_i / T = 1 - 1 = 0$$

This means that we cannot obtain estimates for all coefficients simultaneously. We have to choose one of them arbitrarily.

Now, since (2.1) gives:

$$(2.4) \qquad Y_{ts} = a_o + k_s + u_{ts}$$

we can fix a_o arbitrarily, and obtain the corresponding estimates of the k's. To remove this arbitrariness, we add another suitable relation between the parameters. The one which would recommend itself is that the annual effect of the season would be zero. This would enable us to pass from the sub-annual relation to the annual one by summing up the seasonal effects to zero. Hence we take:

$$\sum_i n_r k_r = 0$$

It follows that $a_o = \overline{Y}$, namely the average over all points of observation. Our model becomes, therefore:

$$(2.5) \qquad y_{ts} = k_1 v_1 + k_2 v_2 + \dots + k_m v_m + u_{ts}$$

where, $y_{ts} = Y_{ts} - \overline{Y}$. In this case we have:

$$(2.6) \qquad \mathbf{M}_{vv} = \frac{1}{T}\left[n_i \ \delta_{ij}\right] \quad \mathbf{M}_{vv}^{-1} = T\left[\delta_{ij} / n_i\right]$$

On the other hand, if $\overline{y}_i$ is the average of the deviations of Y_{ti}, *i.e.*, of the observations in the i-th segment, from their average $\overline{Y}_i$, then:

$$(2.7) \qquad m_{yv_i} = n_i \cdot \overline{y}_i / T$$

If we decide to estimate all the parameters simultaneously we should obtain the same estimates as those given by (2.10). For if we define :

$$(2.11) \qquad \hat{b} = (\hat{a} \quad \hat{k}) \text{ and } X' = [Z' \quad V']$$

then,

$$M_{xx} = \frac{1}{T} X. X' = \begin{bmatrix} M_{zz} & M_{zv} \\ M_{vz} & M_{vv} \end{bmatrix}$$

Denoting the inverse M_{xx}^{-1} by K_{xx}, we can partition it in the same manner and obtain for the submatrices in the first column ;

$$K_{zz} = {}^{(v)}M_{zz}^{-1}, \qquad K_{vz} = - M_{vv}^{-1} . M_{vz} {}^{(v)}M_{zz}^{-1}$$

where ${}^{(v)}M_{zz}$ is defined according to (2.9). Hence :

$$\hat{a} = m_{yx} K_{xz} = m_{yz} {}^{(v)}M_{zz}^{-1} - m_{yv} M_{vv}^{-1} M_{vz} {}^{(v)}M_{zz}^{-1}$$

$$= {}^{(v)}m_{yz} {}^{(v)}M_{zz}^{-1}$$

exactly as in (2.10)

To conclude the discussion, we can say that the results of using formulation (1.2) and (1.5) are the same as those of the classical methods. We have to bear in mind the nonsingularities involved, and the exact number of degrees of freedom lost. On the other hand, if the linear combination $\Sigma a_i z_{its}$ is sufficient to explain "trend" in Y, we can ignore trend terms. If this is not true, we can include them, and the estimation equations will remain formally the same. The use of adjusted series might help to reduce the burden of computations, but it should not conceal the loss of degrees of freedom. On the other hand, if the seasonal parameter estimates turn to be insingnificantly different from zero, we might conclude that Y adjusts itself to changes in z, whether these changes are of a permanent nature or are simply its inverse can be partitioned as follows:

$$(3.4)\quad M_{zz} = \begin{bmatrix} M_{11} & O & \ldots\ldots & O \\ O & M_{22} & \ldots\ldots & O \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ O & O & \ldots\ldots & M_{mm} \end{bmatrix} \text{ and } M_{zz}^{-1} = \begin{bmatrix} M_{11}^{-1} & O & \ldots\ldots & O \\ O & M_{22}^{-1} & \ldots & O \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ O & O & \ldots\ldots & M_{mm}^{-1} \end{bmatrix}$$

Thus the usual least-squares formulae show that the estimates of the c's are those which can be obtained by applying least-squares to each of the m equations (3.2) one at a time. The residual variance, however, is estimated here as the sum squares of residuals over all the T observations points, divided by the total number of degrees of freedom, $T - m\,(p+1)$. Clearly this requires that $T > m\,(p+1)$. The covariance matrices of the estimates differ by a scalar multiplier from those obtained for the r-th equation in isolation, owing to the pooling of the residual variances.

The relation between this formulation and classical methods is not quite evident as in the previous case. However, one can argue as follows. Let there be some seasonal indices, d_{yr} and d_{ir} for the variables Y and z_i in the r-th season. If we denote to seasonally adjusted variables by Y^* and z_i^*, then the observed values in the r-th eeasod would be[4] $Y = d_{yr}\,Y^*$, and $z_i = d_{ir}\,z^*$. The economic relation might be considered as unique for all seasons if it were constructed in terms of the deseasonalized variables. Thus:

$$(3.5)\qquad Y^*_{ts} = \sum_i a_i\, z^*_{its} + u^*_{ts}$$

(4) We can consider $d_{os} = 1$, to allow for the fact that $z_{ost} = 1$ always.

If on the other hand we assume that w_{ts} possess the properties of the the u's (viz., they have constant variance), then we come back to the same position as that given by (2.5) and (2.8).

This simple example suggests that formula (3.2) can be considered in a sense as an extension of the simple additive case, where the *season affects all structural parameters additively.* A change in the specifications of the properties of the disturbances would alter the results considerably. In particular, the estimation of the parameters in (3.8) by least-squares directly would introduce certain complications which will not be considered here. A discussion of the case where u^* ls of a constant variance would lead us away from the original specification given by Klein. But one should be aware of the implications suggested by such an assumption with respect to the residuals in both (3.5) and (3.6).

4.—*The Restricted Multiplicative Case :*

This case, which is represented by (3.2) and (3.3), raises new difficulties. The restrictions given by (3.3) show that we cannot apply least-squares directly. Therefore we shall write down the logarithmic likelihood function (L. L. F.) and obtain maximum likelihood estimates. We shall assume (without loss of generality) that $T = nm$, where n is an integer. The L. L. F. is:

$$(4.1) \qquad L = \text{constant} - \frac{1}{2} \log \sigma^2 - \left(\frac{1}{2\sigma^2} \right) \cdot Q$$

where,

$$Q = \frac{1}{T} \sum_t \sum_s \left(Y_{ts} \quad \sum_i a_i k_s z_{its} \right)^2$$

Let,

$$(4.2) \qquad m_{yi}^{(s)} = \frac{1}{T} \sum_t Y_{ts} z_{its}, \qquad m_{ij}^{(s)} = \frac{1}{T} \sum_t z_{its} z_{jts}$$

We can express first and second derivatives of L as follows:

$$\frac{\partial L}{\partial \mathbf{k}'} = \frac{1}{Q}\mathbf{g}' \qquad \frac{\partial L}{\partial \mathbf{a}} = \frac{1}{Q}\mathbf{h}$$

$$\frac{\partial^2 L}{\partial \mathbf{k}' \partial \mathbf{k}} = \frac{1}{Q}(-\mathbf{F}) + \frac{2}{Q^2}\mathbf{g}'\mathbf{g} \qquad \frac{\partial^2 L}{\partial \mathbf{a}' \partial \mathbf{a}} = \frac{1}{Q}\left[-\Sigma k_s^2 \mathbf{M}_{zz}^{(s)}\right]$$

$$+\frac{2}{Q^2}\mathbf{g}'\mathbf{h} \qquad \frac{\partial^2 L}{\partial \mathbf{k}' \partial \mathbf{a}} = \frac{1}{Q}(\mathbf{H} - 2\mathbf{C}) + \frac{2}{Q^2}\mathbf{g}'\mathbf{h}$$

Accordingly, we can obtain the required coefficients cstimates from:

(4.9) $\quad \mathbf{H}\hat{\mathbf{a}}' - \mathbf{F}\hat{\mathbf{k}}' = \mathbf{O}' \qquad \therefore \quad \hat{\mathbf{k}}' = \mathbf{F}^{-1}\,\mathbf{H}\hat{\mathbf{a}}'$

(4.10) $\quad \hat{\mathbf{k}}\mathbf{H} - \sum_s \hat{k}_s^2\, \hat{\mathbf{a}}\, \mathbf{M}_{zz}^{(s)} = \mathbf{O} \qquad \therefore \quad \sum_s \hat{k}_s^2\, \hat{\mathbf{a}}\, \mathbf{M}_{zz}^{(s)} = \sum_s \hat{k}_s\, \mathbf{m}_{yz}^{(s)}$

This means that,

$$\hat{k}_s = \frac{1}{\hat{\mathbf{a}}\,\mathbf{M}^{(s)}\,\hat{\mathbf{a}}'} \sum_j \left(m_{yj}^{(s)}\, \acute{a}_j\right)$$

We introduce a new set of variables dariables defined as follows:

(4.11) $\quad x_{its} = \hat{k}_s\, z_{its} \qquad (i = 0, 1, \ldots, p)$

Hence,

(4.12) $\quad \mathbf{m}_{yx}^{(s)} = \hat{k}_s\, \mathbf{m}_{yz}^{(s)} \qquad \mathbf{M}_{xx}^{(s)} = \hat{k}^2\, \mathbf{M}_{zz}^{(s)}$

Further,

(4.13) $\quad \mathbf{m}_{yx} = \sum_s \hat{k}_s\, \mathbf{m}_{yz}^{(s)} \qquad \mathbf{M}_{xx} = \Sigma\, \hat{k}_s^2\, \mathbf{M}_{zz}^{(s)}$

we use the well-known maximum-likelihood property, namely that for large T, the variables $T^{\frac{1}{2}}\ (\hat{p}_i - p_i)$ are distributed asymptotically according to a multivariate normal distribution with zero means and a variance-covariance matrix $\mathbf{V}_{pp}$ given by:

$$(4.18) \qquad T.\ V_{pp} = \left[\frac{\partial^2 L}{\partial\, p_i\ \partial\, p_j} \right]^{-1}$$

The estimate $\hat{\mathbf{V}}_{pp}$ of $\mathbf{V}_{pp}$ is given by substituting in (4.18) the maximum—likelihood estimates of the parameters, so that we can express the variance matrix as a function of the observations.

It is clear that by the time we reach the last round of iteration, we would possess estimates of the matrix $\mathbf{M}_{xx}$ and its inverse $\mathbf{M}_{xx}^{-1}$. Substituting the estimates of the parameters in (4.4) we obtain:

$$\hat{\mathbf{F}} = \left[\ \delta_{rs}\ \hat{\mathbf{a}}\ \mathbf{M}^{(s)}_{zz}\ \hat{\mathbf{a}}\ \ \mathbf{A}\ \right] \qquad \hat{\mathbf{C}} = \left[\ \hat{k}_s\ \ \sum_j m^{(s)}_{ij}\ \hat{a}_j \right]$$

We then define the matrices:

$$\mathbf{L} = 2\ \hat{\mathbf{C}} - \mathbf{H} \qquad\qquad \mathbf{K} = \mathbf{F} - \mathbf{L}\mathbf{M}_{xx}^{-1}\ \mathbf{L}'$$

The inverse $\mathbf{K}^{-1}$ can be obtained by well-known methods. The variance-covariance matrix of the a's will then be:

$$(4.19) \qquad \hat{\mathbf{V}}_{aa} = \hat{Q}\left(\mathbf{M}_{xx}^{-1} + \mathbf{M}_{xx}^{-1}\,.\ \mathbf{L}'.\ \mathbf{K}^{-1}.\ \mathbf{L}.\ \mathbf{M}_{xx}^{-1}\right)$$

where,

$$\hat{Q} = m_{yy} - 2\ \hat{k}\mathbf{H}'\hat{a} + \hat{k}\hat{\mathbf{F}}\hat{k}' \quad .$$

$$= m_{yy} - 2\ m_{yx}\ \hat{a}' + \hat{a}\mathbf{M}_{xx}\ \hat{a}'$$

i. e.,

On the other hand, the unrestricted multiplicative case is in many respects useful in this field. First, it shows us how we can combine two distinctly different periods, thus allowing us to retain a fairly acceptable number of degrees of freedom. Further it shows that we can use a simple formula to prove the fact that if an equation has assumed different parameter levels at two periods, then a proper test of significance is available in a simple manner. The method gives us directly the estimates required to test the differences between the two sets of the parameters estimates taken for the same equation at two different periods or from two different samples. Thus, if the underlying regression equation is:

$$(5.1) \qquad Y_t = a_o + \sum_i a_i Z_{it} + u_t$$

we define a variable v_t such that:

$$V_t = 0, \text{ for } t = 1, 2, \ldots, T', \qquad V_t = 1, \text{ for } t = T' + 1, \ldots, T$$

The regression equation becomes:

$$(5.2) \qquad Y_t = (a_o + k_o v_t) + \sum_i (a_i + k_i v_t) Z_{it} + u_t$$

which means that:

(5.1) holds for $t = 1, 2, \ldots, T'$, and

$$(5.3) \qquad Y_t = (a_o + k_o) + \sum_i (a_i + k_i) Z_{it} + u_t \text{ (for } t = T' + 1, \ldots, T)$$

It the additive case, we have only shifts in the constant term, while all $k_i = 0$. In the multiplicative case, we can estimate the coefficients directly using (5.1) and (5.3) for the two periods. The residual variance is estimated by the sumsquare of all residuals divided by the total number of degrees of freedom, viz., $T - 2(p+1)$. Apart from this, the variances of the coefficients' estimates in either case will be proportinal to what obtains directly from simple least squares. All what has been said in dealing with seasonal variations, applies here, with obvious modifications.